Study on Multi Server Markov Modulated queue with finite capacity

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Abstract—Markov modulated queuing models are those in which the primary arrival and service mechanism are influenced by change of phase in secondary Markov process. In this paper we study Markov Modulated Poisson Process with multi server queue with finite capacity and we provide the exact computable matrix expression of the rate matrix that can be used to obtain the performance measures.

Index Terms— Markov arrival process, Markov Modulated Poisson Process, Matrix analytic method, Quasi Birth Death process

1.Introduction

Markovian arrival processesMAP) were introduced by Neuts in 1979 (Neuts 1979) and have been usedextensively in the stochastic modeling such as queueing theory, inventory, reliability, risk, and communications systems.Markovian arrival processes have gained its popularity as it comes from i) its versatility in modeling stochastic systems; ii) its Markovian property that leads to Markovian structures; and iii) the maneuverability in the resulting Markov chains. To learn and use Markovian arrival processes, the only need of basic knowledge of exponential distributions and Poisson processes.

In stochastic processes, the counting process has gained its importance in the field of science and engineering. A number of counting processes have been introduced in order to capture the characteristics of real stochastic processes,. Some well-known counting processes are Poisson processes, compound Poisson processes, Markov modulated Poisson processes, renewal processes, and semi-Markov processes.Markovian arrival processes are considered as generalizations of Poisson processes, compound Poisson processes, and Markov modulated Poisson processes (Neuts 1979). Since their introduction, Markovianarrival processes have been used in the study of various queueing*models.*

The MMPP is most frequently seen in queuing theory (Du 1995; Olivier and Walrand 1994) but it has other interesting applications.There exists a number of papers (e.g. Meier, K - Hellstern (1989), Scott and Smyth (2003)) where Markovian arrival processes and, specifically, superposition of MMPPs are used as a very versatile tool to model variable packet traffic exhibiting long-range dependence.For a good survey on MMPP we refer the reader toFischer, W. and Meier-Hellstern, K(1992).

In this paper we study Markov Modulated Poisson Process with multi server queue with finite capacity and we provide the exact computable matrix expression of the rate matrix that can be used to obtain the performance measures.

2. Preliminaries

The MMPP is a Poisson Process whose rate varies according to a finite Markov chain serving as a random environment. Let Qbe its underlying infinitesimal generator. The arrival rate is $\lambda_i > 0$ when the random environmental state is i. Then, theMMPP is aMAP with $D_0 = Q - \Lambda$and $D_1 = \Lambda$,where $\Lambda = diag(\lambda_1 .., \lambda_m)$.

An $MMPP$ is a doubly stochastic Poisson process described with two parameter matrices D_0 and D_1 of dimension m. The transition probability matrix is given by

$$\int_0^x e^{D_0 t} dt D_1 = [I - e^{D_0 x}]\,(-D_0)^{-1} D_1,\ x \geq 0$$

The generator is given by $Q^* = D_0 + D_1$.

Briefly, the Kronecker sum of two matrices A and B of orders m and $n$is given by

$$A \oplus B = A \otimes I_m + I_n \otimes B$$

Which is of order $mn \times mn$where the matrices I_m and I_n are the identity matrices of order m and n respectively. The Kronecker product of two matrices A and B of order $m \times n$ and $r \times s$ is given by $A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mnB} \end{bmatrix}$ which is of order $mr \times ns$. If A and B are two square matrices of order$m$ and n then $A \otimes B$ is a matrix of mn.

3. Model description

In this paper, we study the model $MMPP/M/c/c+N$ queues with finite capacity. Customer arrive at the system from source $i, i \geq 1$ is an independent $MMPP$. When customer arrives at the system one of the following things may happen

i) If there is available server in the system, the server is on provides service to the customer with probability p; otherwise the server will be in off state with probability $q = 1 - p$ if there is no customer to be served.

ii) If the arriving customer finds all the server busy, it enters into the waiting area with waiting capacity of N customers; otherwise the arriving customer is lost.

The queues $MMPP/M/c/c+N$ can be parameterized by

a. $\lambda$the arrival rate of the exogenously originating calls;
b. $\mu$the service rate, corresponding to the inverse of the mean call holding time;
c. c the number of servers;
d. $N$the queueing capacity;
e. m the number of busy servers in on state
f. $(Q_i, \Lambda_i), 1 \leq i \leq m$ the parameters of busy servers where
$Q_i = \begin{pmatrix} -c_{i1} & c_{i1} \\ c_{i2} & -c_{i2} \end{pmatrix}$, $\Lambda_i = \begin{pmatrix} \lambda_i & 0 \\ 0 & 0 \end{pmatrix}$

The input process to the queue is then an $MMPP$ given by

$Q = Q_1 \oplus Q_2 \oplus Q_3 \dots \oplus Q_m$, $\Lambda = \lambda I + \Lambda_1 \oplus \Lambda_2 \oplus \dots \oplus \Lambda_m$

where$I$ is the identity matrix of order 2^m.

4. Quasi Birth Death process

The state of the system at time t is described as bivariate process

$$\{X(t): t \geq 0\} = \{N(t), I(t), J(t): t \geq 0\}$$

where$N(t)$ denotes the number of customers in the system,I(t) is the total number of busy servers and waiting position occupied and $J(t)$ the state of the Markov process with infinitesimal generator Q. The process $X(t)$ form a continuous Time Markov Chain (CTMC) on the state space

$$S = Z_+ \times \{0,1,2,\dots,c,c+1,\dots,c+N\} \times 2^m.$$

The infinitesimal generator is given by

$$\bar{Q} = \begin{bmatrix} Q-\Lambda & \Lambda & & & & & \\ \mu I & Q-\Lambda-\mu I & \Lambda & & & & \\ & 2\mu I & Q-\Lambda-2\mu I & \Lambda & & & \\ & & 3\mu I & Q-\Lambda-3\mu I & \Lambda & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & & & \\ & & & & & c-\mu I & Q-c\mu I \end{bmatrix}$$

The steady state vector of $\bar{Q}$ is denoted by $\pi = (\pi_0, \pi_1, \dots, \pi_{c+N}$ and satisfies the equations

$$\pi_0(Q - \Lambda) + \pi_1 \mu I = 0$$

$$\pi_{i-1}\Lambda + \pi_i(Q - \Lambda - i\mu I) + \pi_{i+1}(i+1)\mu I = 0, \quad 1 \leq i \leq c-1$$

$$\pi_{i-1}\Lambda + \pi_i(Q - \Lambda - c\mu I) + \pi_{i+1}c\mu I = 0, \quad c \leq i \leq c+N-1$$

$$\pi_{c+N-1}\Lambda + \pi_{c+N}(Q - c\mu I) = 0.$$

subject to the condition $\sum_{i=0}^{c+N} \pi_i e = 1$.

Let $\pi = (\pi_0, \pi_1, \dots, \pi_{c+N})$ denote the steady state probability vector. The vector π is computed by solving the following equations

$$\pi_0 = \pi_1 \mu I[\Lambda - Q]^{-1}$$

$$\pi_i = [\pi_{i+1}(i+1)\mu I + \pi_{i-1}\Lambda][i\mu I + \Lambda - Q]^{-1}, \quad 1 \le i \le c-1$$

$$\pi_i = [\pi_{i+1}c\mu I + \pi_{i-1}\Lambda][c\mu I - Q]^{-1}, \quad c \le i \le c+N-1$$

$$\pi_{c+N} = \pi_{c+N-1}\Lambda[c\mu I - Q]^{-1}$$

with $\sum_{i=0}^{c+N} \pi_i e = 1$.

The rate matrix $\hat{R}(i)$ can be computed on setting $\hat{R}(i) = i\mu[i\mu I + \Lambda - Q]^{-1}$for $1 \le i \le c$ and $\gamma = \frac{1}{c\mu}\pi_{c+N}\Lambda$

we obtain $\pi_0 = c\gamma \hat{R}^{N+1}(c)\prod_{j=1}^{c-1}\hat{R}(c-j)\,[-\mu Q]^{-1}$

$\pi_i = \frac{c}{i}\gamma \hat{R}^{N+1}(c)\prod_{j=1}^{c-i}\hat{R}(c-j)$for $1 \le i \le c-1$

$\pi_i = c\gamma \hat{R}^{N+c-i+1}(c)forc \le i \le c+N$

The constant γ is computed by the normalizing equations.

5. Conclusion

In this study we have provided exact, computable matrix expression of the rate matrix which is the main ingredient for discussion of the qualitative behavior y. The future work of this study is to obtain the main performance measures of the system.

References

[1] Neuts, M. F. *Matrix-geometric Solutions in Stochastic Models: An Algorithmic Approach*. Baltimore:The Johns Hopkins University Press, 1981.

[2] Neuts, M.F. A versatile Markovian point process. Journal of AppliedProbability, 16: 764-779, 1979.

[3] Du, Q. A monotonicity result for a single-server queue subject to a Markov-modulated Poisson process, *J. Appl. Probability* **32**, 1103–1111, 1995

[4] Olivier, C. and Walrand, J. On the existence of finite-dimensional filters for Markov-modulated traffic.*J. Appl. Probability* **31**, 515–525, 1994.

[5] Latouche, G. and Ramaswami, V, *Introduction toMatrixAnalyticMethods in Stochastic Modelling*. Philadelphia: ASA-SIAM, 1999.

[6] Scott, S.L., Smyth, P. The Markov Modulated Poisson Process and Markov Poisson Cascade withApplications to Web Traffic Modeling. Bayesian Statistics 7. *Oxford University Press*, 2003.

[7] Meier, K -Hellstern. The Analysis of a queue arising in overflow models, IEEE Trans. Commun, 37, 367-372, 1989.

[8] Fischer, W. and Meier-Hellstern, K. The Markov-modulated Poisson process (MMPP) cook book. Performance Evaluation 18, No.2, 149-171, 1992.

[9] Breuer, L. and Baum, D. *An Introduction to Queueing Theory and Matrix-Analytic Methods*. Dordrecht, Springer, 2005.